Unitarizable representations of the deformed para-Bose superalgebra $U_q[osp(1/2)]$ at roots of 1

1995 J. Phys. A: Math. Gen. 28 7275

(http://iopscience.iop.org/0305-4470/28/24/023)

J. Phys. A: Math. Gen. 28 (1995) 7275–7285. Printed in the UK

Unitarizable representations of the deformed para-Bose superalgebra $U_q[osp(1/2)]$ at roots of 1

T D Palev† and N I Stoilova†

International Centre for Theoretical Physics, 34100 Trieste, Italy

Received 13 June 1995

Abstract. The unitarizable irreps of the deformed para-Bose superalgebra pB_q, which is isomorphic to $U_q[osp(1/2)]$, are classified at q being root of 1. New finite-dimensional irreps of $U_q[osp(1/2)]$ are found. Explicit expressions for the matrix elements are written down.

1. Introduction

In the present paper we study unitarizable root of unity representations of the Hopf algebra $pB_q(1) \equiv pB_q$, introduced in [1]. It is generated (essentially) by one pair of deformed para-Bose operators $a^{\pm}$. The irreps of pB_q at generic values of q are infinite-dimensional and are realized in deformed para-Bose (pB) Fock spaces $F(p)$, $p \in \mathbb{C}$ [1]. The multimode Hopf algebra $pB_q(n)$, corresponding to n pairs of deformed pB operators $a_1^{\pm}, a_2^{\pm}, \ldots, a_n^{\pm}$ was defined in [2–5]. The case of any number of deformed para-Fermi operators was worked out in [6].

So far various deformations of para-Bose and para-Fermi statistics have been considered from different points of view [7–22]. Some of them are not related to any Hopf algebra structure. A guiding principle of the approaches in [1–5], which we follow, is to preserve, similar to the non-deformed case [23], the identification of $pB_q(n)$ with $U_q[osp(1/2n)]$: $pB_q(n)$ is an associative superalgebra isomorphic (as a Hopf algebra) to the deformed universal enveloping algebra $U_q[osp(1/2n)]$ of the orthosymplectic Lie superalgebra $osp(1/2n)$.

The Hopf algebra structure of $pB_q(n)$ has an important advantage: using the comultiplication, one can define new representations of the deformed operators (and hence of $U_q[osp(1/2n)]$) in any tensor product of representation spaces. In particular one can use the Fock space of n pairs of commuting deformed Bose operators [24–27], since they give a representation of $U_q[osp(1/2n)]$ [28]. Even in the non-deformed case the only effective technique for constructing representations of parabosons or of $osp(1/2n)$ (for large n) is through tensor products of bosonic Fock spaces (see [29] for more disscusions in this respect).

The definition of $U_q[osp(1/2n)]$ in terms of its Chevalley generators is well known [30–35]. Although for $n > 1$ the deformed pB operators $a_1^{\pm}, a_2^{\pm}, \ldots, a_n^{\pm}$ are very different from the Chevalley generators, the relations determining $U_q[osp(1/2n)]$ through $a_1^{\pm}, a_2^{\pm}, \ldots, a_n^{\pm}$

† Permanent address: Institute for Nuclear Research and Nuclear Energy, 1784 Sofia, Bulgaria; e-mail: palev@bgearn.bitnet, stoilova@bgearn.bitnet

0305-4470/95/247275+11$19.50

are not more involved [4, 5]. At $n = 1$, namely in the case we consider, $a^{\pm}$ are proportional to the Chevalley generators of $pB_q = U_q[osp(1/2)]$.

The finite-dimensional irreps of $U_q[osp(1/2)]$ at generic q were constructed in [36, 37]. Some root of unity highest weight irreps were also obtained in [37]; both highest weight and cyclic representations were studied in detail in [38–40].

Our carrier representation spaces $F(p)$, $p \in \mathbb{C}$ will be deformed Fock spaces [1], which are in fact the Verma modules used in [39]. For root of unity cases each such space is no more irreducible; it contains infinitely many invariant subspaces. The irreps are realized in appropriate factor spaces of $F(p)$ with the vacuum being the highest weight vector.

To our best knowledge the root of 1 irreps of $U_q[osp(1/2)]$ obtained in section 3.2 and those labelled with an integer p (sections 3.1.1, 3.1.2, 3.3) have not been described in the literature so far. Our other main result is the classification of the unitarizable Fock irreps of pB_q (= unitarizable Verma representations of $U_q[osp(1/2)]$) at roots of 1 (see (4.2)). We write down explicit expressions for the transformation of the basis under the action of the deformed pB generators.

The reason to pay special attention to the unitarizable representations stems from physical considerations. In all applications of deformed parastatistics known to us [7–22] it is assumed that the Hermitian conjugate $(a^-)^\dagger$ of the annihilation operator a^- equals the creation operator a^+:

$$(a^-)^\dagger = a^+. \tag{1.1}$$

In the case of deformed para-oscillators [9, 15], for instance, or more generally in any deformed quantum mechanics (see for instance [41] and the references therein) the unitarity condition (1.1) is equivalent (as in the canonical case) to the requirement that the position and the momentum operators be self-adjoint operators.

The paper is organized as follows. In section 2 we recall the definition of the deformed para-Bose algebra and its Fock representations at generic q. Section 3 is devoted to a detailed study of the root of 1 irreps. The non-decomposable representations both finite-dimensional and infinite-dimensional are also mentioned. The unitarizable representations are classified in section 4. Section 5 contains some concluding remarks.

Throughout we use the following abbreviations and notation: $\mathbb{C}$, all complex numbers; $\mathbb{Z}$, all integers; $\mathbb{Z}_+$, all non-negative integers; $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, the ring of all integers modulo 2; $[A, B] = AB - BA$, $\{A, B\} = AB + BA$.

2. The para-Bose Hopf algebra pB_q and its Fock representations

To begin with we summarize some of the results from [1], slightly changing the notation.

Definition 1. The para-Bose algebra pB_q, $q \in \mathbb{C}\backslash\{0, \pm 1\}$, is the associative superalgebra over $\mathbb{C}$ with unit 1 defined by the following generators and relations:

$$\text{Generators: } a^{\pm},\ K^{\pm 1} \tag{2.1}$$

$$\text{Relations: } KK^{-1} = K^{-1}K = 1 \qquad Ka^{\pm} = q^{\pm 2}a^{\pm}K \qquad \{a^+, a^-\} = \frac{K - K^{-1}}{q - q^{-1}} \tag{2.2}$$

$$\mathbb{Z}_2\text{-grading: } \deg(K^{\pm 1}) = \bar{0} \qquad \deg(a^{\pm}) = \bar{1}. \tag{2.3}$$

Setting $K = q^H$ with $q = e^{\eta}$, $\eta \in \mathbb{C}$, one recovers as $q \to 1$ ($\eta \to 0$) the defining relations of the non-deformed para-Bose operators [42] $[\{a^{\xi}, a^{\eta}\}, a^{\epsilon}] = (\epsilon - \eta)a^{\xi} + (\epsilon - \xi)a^{\eta}$ with $H = \{a^+, a^-\}$ and $\xi, \eta, \epsilon = \pm$ or ± 1.

It has already been shown in [1] how pB_q can be endowed with a comultiplication, a counity and an antipode; here we shall be not concerned with this additional structure.

The (deformed) Fock space $F(p)$ is defined for any $p \in \mathbb{C}$, postulating that $F(p)$ contains a vacuum vector $|p;0\rangle$, namely

$$a^-|p;0\rangle = 0 \qquad K|p;0\rangle = q^p|p;0\rangle \qquad (\Leftrightarrow H|p;0\rangle = p|p;0\rangle). \tag{2.4}$$

At the limit $q \to 1$ the above definition of $F(p)$ reduces to the usual one ($a^-|p;0\rangle = 0, a^-a^+|p;0\rangle = p|p;0\rangle$), where p is the order of the parastatistics [42].

$F(p)$ is an infinite-dimensional linear space with a basis

$$|p;n\rangle = (a^+)^n|p;0\rangle \qquad n \in \mathbb{Z}_+. \tag{2.5}$$

Let

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad \{n\} = \frac{q^n + q^{-n}}{q + q^{-1}}. \tag{2.6}$$

The transformation of the basis (2.5) follows from (2.4). The relations below are written in a slightly more general form in order to accommodate also all root of 1 cases. For

$$|p;0\rangle, |p;1\rangle, \ldots, |p;L\rangle \tag{2.7a}$$

set

$$K|p;n\rangle = q^{2n+p}|p;n\rangle \tag{2.7b}$$

$$a^-|p;n\rangle = [n]\{n+p-1\}|p;n-1\rangle \qquad \text{for } n = \text{even number} \tag{2.7c}$$

$$a^-|p;n\rangle = [n+p-1]\{n\}|p;n-1\rangle \qquad \text{for } n = \text{odd number} \tag{2.7d}$$

$$a^+|p;n\rangle = |p;n+1\rangle \qquad n \neq L \tag{2.7e}$$

$$a^+|p;L\rangle = 0. \tag{2.7f}$$

The transformations of the basis (2.5) are given with (2.7a)–(2.7e) when $L = \infty$. At generic q each Fock space $F(p)$ is an infinite-dimensional simple (= irreducible) pB_q module [1].

3. Root of unity representations

If q is a root of 1 the pB_q module $F(p)$ may no longer be irreducible. More precisely,

Proposition 1. The Fock space $F(p)$ is non-decomposable if and only if $q = \mathrm{e}^{\frac{1}{2}\mathrm{i}\pi m/k}$ for every $m, k \in \mathbb{Z}$ such that $q \notin \{\pm 1, \pm \mathrm{i}\}$, i.e. $m \neq 0 (\mathrm{mod}\, k)$.

Proof. We exclude from consideration $q \in \{\pm 1, \pm \mathrm{i}\}$ since at these values of q the expressions (2.7) are not defined (at $q = \pm 1$ also pB_q is undefined).

If $q \neq \mathrm{e}^{\frac{1}{2}\mathrm{i}\pi m/k}$ the coefficients in front of $|p;n\rangle$ in (2.7c, d) vanish only for $n = 0$. Hence the only singular vector is the vacuum, i.e. $F(p)$ is an irreducible module. If $q = \mathrm{e}^{\frac{1}{2}\mathrm{i}\pi m/k}$ then, for instance, $|p;2k\rangle$ is a singular vector, $a^-|p;2k\rangle = 0$. Therefore the proper subspace of $F(p)$, spanned on $|p;n\rangle, n \geqslant 2k$ is an invariant subspace. Then equation (2.7e) yields that the representation is non-decomposable. This completes the proof. □

The algebras pB_q corresponding to all possible values of m and k contain several isomorphic copies. Clearly we can always assume that $k > 0$ and that m and k are co-prime, i.e. m/k is an irreducible fraction. Further we note that the algebras pB_q and $pB_{\tilde{q}^\xi}$ are isomorphic for $q = e^{\frac{1}{2}\mathrm{i}\pi m/k}$ and $\tilde{q}^\xi = \mathrm{e}^{\frac{1}{2}\mathrm{i}\pi(2k+\xi m)/k}$ whenever $\xi = +$ and $m = 1, 2, \ldots, 2k-1$ or $\xi = -$ and $m = 1, 2, \ldots, k-1$, since the generators $\tilde{a}^\pm = a^{\pm\xi}$ and $\tilde{K} = -\xi K$ of $pB_{\tilde{q}^\xi}$ satisfy the defining relations (2.1)–(2.3) for pB_q. The case $\xi = +$

indicates that we can set $m \in \{1, 2, \ldots, 2k-1\}$; the case $\xi = -$ further shows that without loss of generality we can assume that $m \in \{1, 2, \ldots, k-1\}$. The case $k = 1$ is excluded from these conditions. Thus, without losing any of the algebras pB_q for which the Fock space $F(p)$ is non-decomposable, we restrict m and k to values which we call admissible. The fraction m/k is said to be admissible if

$$\frac{m}{k} \in \left\{\frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}\right\} \qquad m \text{ and } k \text{ are co-prime} \quad k = 2, 3, 4, \ldots. \tag{3.1}$$

Passing to a discussion of the root of 1 representations, we first note that the vectors

$$|p; 0\rangle, |p; 2k\rangle, |p; 4k\rangle, \ldots, |p; 2kN\rangle, \ldots \tag{3.2}$$

are singular vectors in $F(p)$, $a^-|p; 2kN\rangle = 0, N \in \mathbb{Z}_+$. The subspaces $V_{|p;2kN\rangle} = \text{span}\{|p; n\rangle | n \geqslant 2kN\}$ are infinite-dimensional invariant subspaces of $F(p)$ with highest weight vectors $|p; 2kN\rangle$. Clearly

$$F(p) = V_{|p;0\rangle} \supset V_{|p;2k\rangle} \supset V_{|p;4k\rangle} \supset \cdots \supset V_{|p;2kN\rangle} \supset \cdots. \tag{3.3}$$

For each $N < M \in \mathbb{Z}_+$ define a $2(M-N)k$-dimensional factor space $W_{|p;2kN\rangle,M} = V_{|p;2kN\rangle}/V_{|p;2kM\rangle}$. Let ξ_x be the equivalence class of $x \in \xi_x$. The vectors $\xi_{|p;2kN\rangle}, \xi_{|p;2kN+1\rangle}, \xi_{|p;2kN+2\rangle}, \ldots, \xi_{|p;2kM-1\rangle}$ constitute a basis in $W_{|p;2kN\rangle,M}$. The relations $a^{\pm}\xi_{|p;n\rangle} = \xi_{a^{\pm}|p;n\rangle}$, $K\xi_{|p;n\rangle} = \xi_{K|p;n\rangle}$ endow $W_{|p;2kN\rangle,M}$ with a structure of a pB_q module. Observe that $a^+\xi_{|p;2kM-1\rangle} = 0$.

We shall simplify the notation, identifying $\xi_{|p;n\rangle}$ with its representative $|p; n\rangle$. Then equations (2.7) with $L = 2kM - 1$ give the transformations of $W_{|p;2kN\rangle,M}$. If $M - N > 1$, equations (2.7) with $L = 2kM - 1$ define an non-decomposable finite-dimensional representation of pB_q in $W_{|p;2kN\rangle,M}$ for any $p \in \mathbb{C}$. If $M - N = 1$, $W_{|p;2kN\rangle,N+1}$ is either irreducible or non-decomposable.

Proposition 2. For each $s \in \mathbb{Z}$ and $N \in \mathbb{Z}_+$ the pB_q modules $W_{|p;0\rangle,1}$ and $W_{|p+4ks;2kN\rangle,N+1}$ are equivalent; they carry one and the same $2k$-dimensional representation of pB_q.

Proof. The one-to-one linear map $\varphi(|p; n\rangle) = |p+4ks; n+2kN\rangle$, $\quad n = 0, 1, 2, \ldots, 2k-1$ of $W_{|p;0\rangle,1}$ onto $W_{|p+4ks;2kN\rangle,N+1}$ is an intertwining operator, $a\varphi(|p; n\rangle) = \varphi(a|p; n\rangle), a = a^{\pm}, K$. In particular the matrices of the generators $a^{\pm}$ and K are the same in the basis of $W_{|p;0\rangle,1}$ and in the basis of $W_{|p+4ks;2kN\rangle,N+1}$, respectively. □

In view of proposition 2 from now on we shall consider only the vacuum modules $W_{|p;0\rangle,1}$, restricting also the values of p to the interval

$$0 < \text{Re}(p) \leqslant 4k. \tag{3.4}$$

We write $W(|p; m\rangle, |p; n\rangle)$ whenever we wish to indicate that $|p; m\rangle, |p; m+1\rangle, |p; m+2\rangle, \ldots, |p; n\rangle$ is a basis in the linear space $W(|p; m\rangle, |p; n\rangle)$. In view of this $W_{|p;0\rangle,1} = W(|p; 0\rangle, |p; 2k-1\rangle)$.

We proceed to study in detail the structure of the Fock spaces for different admissible values of m/k. To this end we will consider three cases: 3.1, k is even, m is odd; 3.2, k is odd, m is odd; 3.3, k is odd, m is even.

3.1. The case k=even, m=odd

If p is not an integer the coefficients $\{n + p - 1\}$ in (2.7*c*) and $[n + p - 1]$ in (2.7*d*) never vanish. Therefore the vectors (3.2) are the only singular vectors in $F(p)$ and the vacuum $|p; 0\rangle$ is the only singular vector in $W_{|p;0\rangle,1}$. The eigenvalues of K on $|p; 0\rangle$ are different for different p, obeying (3.4). This gives rise to the following

Proposition 3. The pB_q modules $W(|p;0\rangle, |p;2k-1\rangle)$ are simple for $p \notin \{1, 2, \ldots, 4k\}$. All of them are $2k$-dimensional. The irreps corresponding to different p from (3.4) are inequivalent. The transformation of the basis is described with (2.7) for $L = 2k - 1$.

3.1.1. Representations with even p. All modules $W(|p;0\rangle|p;2k-1\rangle)$ corresponding to even values of p are no more irreducible.

Proposition 4. To each $p \in \{2, 4, \ldots, 4k\}$ there corresponds a simple pB_q module $W(|p;0\rangle), |p;L\rangle)$ with a basis $|p;0\rangle, |p;1\rangle, \ldots, |p;L\rangle$ and values of L as follows:

$$L = 2k - p \qquad \text{for } p \in \{2, 4, \ldots, 2k\} \tag{3.5}$$

$$L = 4k - p \qquad \text{for } p \in \{2k+2, 2k+4, \ldots, 4k\}. \tag{3.6}$$

The transformation of the basis is described with the equations (2.7) for the above values of L. All $2k$ modules carry different, inequivalent irreps of pB_q.

Proof. We consider in detail the case $p \in \{2, 4, \ldots, 2k\}$. The module $W(|p;0\rangle, |p;2k-1\rangle)$ contains only two singular vectors $|p;0\rangle$ and $|p;2k-p+1\rangle$, i.e. $a^-|p;0\rangle = 0, a^-|p;2k-p+1\rangle = 0$. In view of (2.7*e*) $W(|p;0\rangle, |p;2k-1\rangle)$ is non-decomposable. Its invariant subspace $W(|p;2k-p+1\rangle, |p;2k-1\rangle)$ is simple. The factor space

$$W(|p;0\rangle, |p;2k-1\rangle)/W(|p;2k-p+1\rangle, |p;2k-1\rangle)$$

with a basis $\xi_{|p;0\rangle}, \xi_{|p;1\rangle}, \ldots, \xi_{|p;2k-p\rangle}$ is turned into an irreducible pB_q module setting $a\xi_{|p;n\rangle} = \xi_{a|p;n\rangle}$ for any $a = a^{\pm}, K$. Therefore $a^+\xi_{|p;2k-p\rangle} = \xi_{|p;2k-p+1\rangle} = 0$. As before we identify the equivalence classes with their representatives: $\xi_{|p;n\rangle} = |p;n\rangle$. Then $W(|p;0\rangle, |p;2k-1\rangle)/W(|p;2k-p+1\rangle, |p;2k-1\rangle) = W(|p;0\rangle, |p;2k-p\rangle)$ and the transformation relations of $W(|p;0\rangle, |p;2k-p\rangle)$ are given with equations (2.7) for $L = 2k-p$. The transformations of the invariant subspaces are described also with (2.7), but for $n = 2k-p+1, 2k-p+2, \ldots, 2k-1 = L$.

The cases with $p \in \{2k+2, 2k+4, \ldots, 4k\}$ are similar. The only singular vectors in $W(|p;0\rangle, |p;2k-1\rangle)$ are the vacuum $|p;0\rangle$ and $|p;4k-p+1\rangle$. Therefore the invariant subspace $W(|p;4k-p+1\rangle, |p;2k-1\rangle)$ is irreducible and its transformations are described with (2.7) for $n = 4k-p+1, 4k-p+2, \ldots, 2k-1 = L$. The factor space $W(|p;0\rangle, |p;4k-p\rangle)$ is also irreducible and transforms according to (2.7) with $L = 4k-p$.

We have four kinds of simple pB_q modules, which are pairwise equivalent:

$$W(|p;2k-p+1\rangle, |p;2k-1\rangle) = W(|p';0\rangle, |p';4k-p'\rangle)$$
$$\text{for } p = 4k - p' + 2 \in \{2, 4, \ldots, 2k\} \tag{3.7}$$

$$W(|p;4k-p+1\rangle, |p;2k-1\rangle) = W(|p';0\rangle, |p';2k-p'\rangle)$$
$$\text{for } p = 4k - p' + 2 \in \{2k+2, 2k+4, \ldots, 4k\}. \tag{3.8}$$

For instance the intertwining operator of $W(|p;2k-p+1\rangle, |p;2k-1\rangle)$ onto $W(|p';0\rangle, |p';4k-p'\rangle)$ reads:

$$\varphi(|p;n\rangle) = |p';n'\rangle$$
$$n = 2k - p + 1, 2k - p + 2 \Longleftrightarrow n' = 0, 1, 2, \ldots, 4k - p'. \tag{3.9}$$

Thus we are left only with vacuum modules. It remains to show that all modules (3.5) and (3.6), i.e.

$$W(|p;0\rangle, |p;2k-p\rangle) \qquad \text{and} \qquad W(|p+2k;0\rangle, |p+2k;2k-p\rangle)$$
$$p \in \{2, 4, \ldots, 2k\} \tag{3.10}$$

are inequivalent. To this end we observe that the modules corresponding to different p in (3.10) have different dimensions. The modules with the same dimensions, namely $W(|p;0\rangle, |p;2k-p\rangle)$ and $W(|p+2k;0\rangle, |p+2k;2k-p\rangle)$ have different spectra of the Cartan generator K and therefore are also inequivalent. This completes the proof. □

It has been noted in [37, 43] that the Casimir operator of $U_q[osp(1/2)]$ is no longer sufficient to label the root of unity representations. In particular this is the case with the modules (3.10), which have the same dimension. The Casimir operator [36] reads in our notation:

$$2C_2(q) = q^2K^2 + q^{-2}K^{-2} + (q^2-q^{-2})(q-q^{-1})(q^2K+q^{-2}K^{-1})a^-a^+ \\ -(q^2-q^{-2})^2(a^-)^2(a^+)^2. \tag{3.11}$$

Its eigenvalue is one and the same, $C_2(q) = \frac{1}{2}(q^{2p-2}+q^{-2p+2})+2$ on four inequivalent modules, namely

$$W(|p;0\rangle, |p;2k-p\rangle) \qquad W(|p+2k;0\rangle, |p+2k;2k-p\rangle) \qquad \dim = 2k-p+1$$
$$W(|2k-p+2;0\rangle, |2k-p+2;p-2\rangle) \qquad W(|4k-p+2;0\rangle, |4k-p+2;p-2\rangle) \\ \dim = p-1. \tag{3.12}$$

Let us add that the additional central elements [37, 38] $(a^\pm)^{2k}$ and $(K)^{2k}$ do not distinguish among the inequivalent modules with the same dimension. In fact the operators $(a^\pm)^{2k}$ vanish within each simple module $W(|p;0\rangle, |p;L\rangle)$.

3.1.2. *Representations with odd p.*

Proposition 5. To each $p \in \{1,3,\ldots,4k-1\}$ there corresponds an irreducible pB_q module $W(|p;0\rangle, |p;L\rangle)$ with a basis $|p;0\rangle, |p;1\rangle, \ldots, |p;L\rangle$, where $L = p(k-1)(\mathrm{mod}\,2k)$. The transformations of the basis are given with equations (2.7) for the corresponding values of L. All these $2k$ modules carry inequivalent irreps of pB_q.

Proof. The proof is similar to that of proposition 4. We stress certain points only. The modules $W(|p;0\rangle, |p;2k-1\rangle)$ with $p = k+1, 3k+1$ are irreducible; the rest of the modules

$$W(|p;0\rangle, |p;2k-1\rangle) \qquad p \in \{1,3,5,\ldots,4k-1\} \qquad p \neq k+1, 3k+1 \tag{3.13}$$

are non-decomposable. Each module in (3.13) contains apart from the vacuum only one more singular vector. Therefore the invariant subspaces and the factor spaces are irreducible. Also here each invariant subspace is equivalent to a factor space. Therefore (up to equivalence) we are left only with the vacuum modules. The modules with the same dimension cannot be separated by the Casimir operator. They have, however, different spectra of the Cartan generator K. Hence they carry inequivalent representations of pB_q.

3.2. *The case k = odd, m = odd*

For any $N \in \mathbb{Z}_+$ (see (2.7)) $a^-|p;kN\rangle = 0$. Therefore the Fock spaces contain more singular vectors than in the general case (see (3.2)). Now

$$|p;0\rangle, |p;k\rangle, |p;2k\rangle, \ldots, |p;kN\rangle, \ldots \tag{3.14}$$

are singular vectors in $F(p)$. For each N the subspace $V_{|p;kN\rangle} = \mathrm{span}\{|p;n\rangle | n \geqslant kN\}$ is an infinite-dimensional non-decomposable invariant subspace of $F(p)$. Because of the inclusions

$$F(p) = V_{|p;0\rangle} \supset V_{|p;k\rangle} \supset V_{|p;2k\rangle} \supset \cdots \supset V_{|p;kN\rangle} \supset \cdots$$

one can build up various non-decomposable finite-dimensional pB_q modules. For each $N < M \in \mathbb{Z}_+$

$$W(|p; kN\rangle, |p; kM-1\rangle) = V_{|p;kN\rangle}/V_{|p;kM\rangle} \tag{3.15}$$

is an $(M-N)k$-dimensional non-decomposable pB_q module with singular vectors $|p;kN\rangle, |p;(k+1)N\rangle, \ldots, |p;k(M-1)\rangle$. As before we do not distinguish between the equivalence classes and their representatives. Since $a^+\xi_{|p;kM-1\rangle} = \xi_{a^+|p;kM-1\rangle} = 0 \Leftrightarrow a^+|p;kM-1\rangle = 0$ the transformation of the factor space (3.15) is given with equations (2.7) for $L = kM-1$. If $M-N=1$, $W(|p;kN\rangle, |p;k(N+1)-1\rangle) = V_{|p;kN\rangle}/V_{|p;k(N+1)\rangle}$ is either irreducible or non-decomposable. We are mainly concerned with the classification of the irreducible pB_q modules. Therefore, as a first step, we identify some equivalent modules. From proposition 2 we know that for a given pB_q algebra (i.e. for a fixed admissible fraction m/k) all irreps can be extracted from the collection of the modules $W(|p;0\rangle, |p;2k-1\rangle)$ with $0 < \mathrm{Re}(p) \leqslant 4k$. According to (3.15) (when $N=0$, $M=2$) the above module is non-decomposable. It contains at least two singular vectors, namely $|p;0\rangle$ and $|p;k\rangle$. The subspace $W(|p;k\rangle, |p;2k-1\rangle)$ is an invariant pB_q subspace and therefore the factor space $W(|p;0\rangle, |p;k-1\rangle) = W(|p;0\rangle, |p;2k-1\rangle)/W(|p;k\rangle, |p;2k-1\rangle)$ is also a pB_q module.

Proposition 6. The collection of all pB_q modules $W(|p;0\rangle, |p;k-1\rangle)$ is equivalent to the collection of all invariant subspaces $W(|p;k\rangle, |p;2k-1\rangle)$ when $0 < \mathrm{Re}(p) \leqslant 4k$. More precisely,

$$W(|p;0\rangle, |p;k-1\rangle) = W(|p+2k;k\rangle, |p+2k;2k-1\rangle) \qquad 0 < \mathrm{Re}(p) \leqslant 2k \tag{3.16a}$$

$$W(|p;0\rangle, |p;k-1\rangle) = W(|p-2k;k\rangle, |p-2k;2k-1\rangle) \qquad 2k < \mathrm{Re}(p) \leqslant 4k. \tag{3.16b}$$

Proof. The transformations of $W(|p;0\rangle, |p;k-1\rangle)$ are given with equations (2.7) for $L = k-1$. The same equations describe the action of the pB_q generators on $W(|p;k\rangle, |p;2k-1\rangle)$ for $L = 2k-1$ and $n = k, k+1, \ldots, 2k-1$. The corresponding intertwining operator φ, defined on the bases, reads:

$$\varphi(|p;n\rangle) = |p+2\xi k; n+k\rangle \qquad n = 0,1,2,\ldots,k-1 \tag{3.17}$$

where $\xi = 1$ corresponds to (3.16a) and $\xi = -1$ to (3.16b).

In view of proposition 6 we shall consider only the vacuum modules $W(|p;0\rangle, |p;k-1\rangle)$, restricting as before the values of p to the interval (3.4).

If p is not an even number the coefficients $\{n+p-1\}$ in (2.7c) and $[n+p-1]$ in (2.7d) never vanish. Therefore the vectors (3.14) are the only singular vectors in $F(p)$ and the vacuum $|p;0\rangle$ is the only singular vector in $W(|p;0\rangle, |p;k-1\rangle)$. In contrast, if p is an even number $W(|p;0\rangle, |p;k-1\rangle)$ contains an extra singular vector. We collect the results in a proposition.

Proposition 7. The pB_q modules $W(|p;0\rangle, |p;k-1\rangle)$ are simple for $p \notin \{2,4,\ldots,4k\}$ with $L = k-1$. To each $p \in \{2,4,\ldots,4k\}$ there corresponds an irreducible pB_q module $W(|p;0\rangle, |p;L\rangle)$ with $L = (k-p)(\mathrm{mod}\, k)$, transformed according to equations (2.7). All these modules carry inequivalent irreps of pB_q.

3.3. The case k = odd, m = even

According to proposition 2 the irreps (up to equivalence) are realized in the vacuum modules $W(|p;0\rangle, |p;2k-1\rangle)$ with $0 < \mathrm{Re}(p) \leqslant 4k$. For the algebras from this class one can further restrict the values of p.

Proposition 8. The following modules are equivalent:

$$W(|p;0\rangle, |p;2k-1\rangle) = W(|p+2k;0\rangle, |p+2k;2k-1\rangle) \qquad \text{for } 0 < \mathrm{Re}(p) \leqslant 2k \tag{3.18}$$

$$W(|p;0\rangle, |p;2k-1\rangle) = W(|p+k;0\rangle, |p+k;2k-1\rangle) \quad \text{for } 0 < \mathrm{Re}(p) \leqslant k \text{ and } m = 4(\mathrm{mod}\,4). \tag{3.19}$$

Proof. The intertwining operators are $\varphi(|p;n\rangle) = |p+\alpha k;n\rangle$ for $n = 0, 1, \ldots, 2k-1$, where $\alpha = 2$ for (3.18) and $\alpha = 1$ for (3.19).

Hence, without loss of generality we assume

$$0 < \mathrm{Re}(p) \leqslant 2k \quad \text{if } m = 2(\mathrm{mod}\,4) \qquad \text{and} \qquad 0 < \mathrm{Re}(p) \leqslant k \quad \text{if } m = 4(\mathrm{mod}\,4). \tag{3.20}$$

Proposition 9. To each $m = 2(\mathrm{mod}\,4)$ with $0 < \mathrm{Re}(p) \leqslant 2k$ and to each $m = 4(\mathrm{mod}\,4)$ with $0 < \mathrm{Re}(p) \leqslant k$ there corresponds an irreducible pB_q module $W(|p;0\rangle, |p;L\rangle)$ with $L = 2k-1$ if p is not an integer and with $L = p(k-1)(\mathrm{mod}\,2k)$ if p is an integer. All such modules are inequivalent. They transform according to (2.7).

We skip the proof.

4. Unitarizable representations

In the present section we classify the unitarizable Fock representations of the deformed para-Bose superalgebra pB_q. The concept of an unitarizable representation of an arbitrary associative algebra A depends on the definition of the antilinear anti-involution $\omega : A \to A$ and on the metric in the corresponding A-module.

Having in mind the physical condition (1.1) and the requirement the 'Hamiltonian' H to be a Hermitian operator, we define ω on the generators $a^{\pm}$ and K as $\omega(a^{\pm}) = a^{\mp}$, $\omega(K^{\pm 1}) = K^{\mp 1}$ and extend it on pB_q as an antilinear anti-involution: $\omega(\alpha a + \beta b) = \alpha^* \omega(a) + \beta^* \omega(b)$, $\omega(ab) = \omega(b)\omega(a)$ for all $a, b \in pB_q$ and $\alpha, \beta \in \mathbb{C}$.

The representation of pB_q in a Hilbert space W with scalar product $(\,,)$ is unitarizable if $\omega(a) = a^\dagger$ for all $a \in pB_q$. On the generators of pB_q the unitarity condition yields $(a^-)^\dagger = a^+, K^\dagger = K^{-1}$. The problem is to select those irreducible modules $W(|p;0\rangle, |p;L\rangle)$ for which the unitarity condition can be satisfied. To this end we introduce a new basis $|p;n) = \alpha(p;n)|p;n\rangle$, $n = 1, 2, \ldots, L, \alpha(p;n) \in \mathbb{C}$ which is declared to be orthonormal. Then the unitarity condition is equivalent to the requirement that the following two equations be satisfied:

$$\left|\frac{\alpha(p;n)}{\alpha(p;n+1)}\right|^2 = \frac{2\sin(\frac{1}{2}\pi(m/k)(n+p))\cos(\frac{1}{2}\pi(m/k)(n+1))}{\sin(\pi m/k)} \qquad n = \text{even} \tag{4.1a}$$

$$\left|\frac{\alpha(p;n)}{\alpha(p;n+1)}\right|^2 = \frac{2\sin(\frac{1}{2}\pi(m/k)(n+1))\cos(\frac{1}{2}\pi(m/k)(n+p))}{\sin(\pi m/k)} \qquad n = \text{odd}. \tag{4.1b}$$

The unknowns in the above equations are the algebras pB_q, i.e. the admissible pairs m/k and the irreducible modules $W(|p;0\rangle, |p;L\rangle)$, i.e., the values of p and L. Equations (4.1) have solutions only if their right-hand side is a non-negative number for any n from the basis in $W(|p;0\rangle, |p;L\rangle)$. Thus, the problem is to solve a set of inequalities. Below we list the algebras pB_q with $q = \mathrm{e}^{\frac{1}{2}\mathrm{i}\pi m/k}$ in terms of the admissible m and k and their unitarizable representations.

$$\begin{array}{lll} \text{The algebra } pB_q & \text{Unitarizable modules} & \\ (1)\ m=1, k=3,5,7,\ldots & W(|p;0\rangle, |p;k-1\rangle)\ 0\leqslant p\leqslant 2 & \\ (2)\ m=1, k=2,4,6,\ldots & W(|p;0\rangle, |p;k-p\rangle)\ p=1,3,5,\ldots,k-1 & \\ (3)\ m=1, k=3,5,7,\ldots & W(|p;0\rangle, |p;k-p\rangle)\ p=2,4,6\ldots,k-1 & (4.2) \\ (4)\ m=1(\mathrm{mod}\,4), k=2,3,4,\ldots & W(|k-1;0\rangle, |k-1;1\rangle) & \\ (5)\ m=3(\mathrm{mod}\,4), k=2,3,4,\ldots & W(|3k-1;0\rangle, |3k-1;1\rangle) & \\ (6)\ m=3, k=10,12,14,\ldots & W(|3k-3;0\rangle, |3k-3;3\rangle). & \end{array}$$

The above equations indicate that the algebras pB_q corresponding to $m=1$ and any odd k have a continuous class of unitarizable representations. In all other cases the number of unitarizable irreps is finite and in fact each algebra with $m\neq 1$ has no more than two representations. In cases (4) and (5) the representation is two-dimensional. In the new basis the matrices of the generators read:

$$\begin{aligned} a^- &= \begin{pmatrix} 0 & \sqrt{\cos\frac{1}{2}\pi(m/k)/\sin\frac{1}{2}\pi(m/k)} \\ 0 & 0 \end{pmatrix} \\ a^+ &= \begin{pmatrix} 0 & 0 \\ \sqrt{\cos\frac{1}{2}\pi(m/k)/\sin\frac{1}{2}\pi(m/k)} & 0 \end{pmatrix} \\ K &= \begin{pmatrix} \mathrm{i}e^{-\frac{1}{2}\mathrm{i}\pi m/k} & 0 \\ 0 & \mathrm{i}e^{\frac{1}{2}\mathrm{i}\pi m/k} \end{pmatrix}. \end{aligned} \tag{4.3}$$

Similarly, the four-dimensional representation from the case (6) reads:

$$\begin{aligned} a^- &= \begin{pmatrix} 0 & \sqrt{\cos(9\pi/2k)/\sin(3\pi/2k)} & 0 & 0 \\ 0 & 0 & \sqrt{2\sin(3\pi/k)} & 0 \\ 0 & 0 & 0 & \sqrt{\cos(9\pi/2k)/\sin(3\pi/2k)} \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ K &= \begin{pmatrix} \mathrm{i}e^{-9\mathrm{i}\pi/2k} & 0 & 0 & 0 \\ 0 & \mathrm{i}e^{-3\mathrm{i}\pi/2k} & 0 & 0 \\ 0 & 0 & \mathrm{i}e^{3\mathrm{i}\pi/2k} & 0 \\ 0 & 0 & 0 & \mathrm{i}e^{9\mathrm{i}\pi/2k} \end{pmatrix} \end{aligned} \tag{4.4}$$

with a^+ represented by the transposed matrix of a^-. Note that the algebras with $m=3$ and $k=10,12,14,\ldots$ have only two unitarizable irreps, namely (4.3) and (4.4); the algebras with $m=3$ and $k=2,4,6,8$ and those with $m=5,7,9,\ldots$ have only a two-dimensional unitarizable irrep. The algebras pB_q with even m have no unitarizable representations at all.

The transformation relations of all unitarizable modules can be written in a compact form. In the orthonormal basis $|p;n\rangle$ equations (2.7) read:

$$K|p;n\rangle = e^{-\frac{1}{2}\mathrm{i}\pi m/k(2n+p)}|p;n\rangle \tag{4.5a}$$

$$a^-|p;n\rangle = \sqrt{\frac{2\sin(\frac{1}{2}\pi(m/k)n)\cos(\frac{1}{2}\pi(m/k)(p+n-1))}{\sin(\pi m/k)}}|p;n-1\rangle \qquad n=\text{even} \tag{4.5b}$$

$$a^-|p;n\rangle = \sqrt{\frac{2\sin(\frac{1}{2}\pi(m/k)(p+n-1))\cos(\frac{1}{2}\pi(m/k)n)}{\sin(\pi m/k)}}|p;n-1\rangle \qquad n=\text{odd} \tag{4.5c}$$

$$a^+|p;n\rangle = \sqrt{\frac{2\sin(\frac{1}{2}\pi(m/k)(p+n))\cos(\frac{1}{2}\pi(m/k)(n+1))}{\sin(\pi m/k)}}|p;n+1\rangle \qquad n=\text{even} \tag{4.5d}$$

$$a^+|p;n) = \sqrt{\frac{2\sin(\frac{1}{2}\pi(m/k)(n+1))\cos(\frac{1}{2}\pi(m/k)(p+n))}{\sin(\pi m/k)}}|p;n+1) \qquad n = \text{odd}. \tag{4.5e}$$

5. Concluding remarks and discussions

We have studied root of unity representations of the deformed para-Bose algebra $pB_q = U_q[osp(1/2)]$ with a particular emphasis on the unitarizable irreps. All of them are realized in finite-dimensional modules with a highest and a lowest weight. The irreps from section 3.2 and also all irreps corresponding to integer p (except $p = k+1, 3k+1$ in section 3.1.2) are new.

In the non-deformed case the representations of the para-Bose operators, corresponding to an order of the statistics $p = 1$ reduce to usual Bose operators [42]. In [1] it was shown that a similar relation holds in the deformed case for generic q. It is straightforward to check that in the cases $p = 1$, $m = 1$, $k = 2, 3, \ldots$ equations (4.5) recover also all root of unity unitarizable irreps of the deformed Bose operators [24–27] as given in [1].

Using the approach of the present paper one can try to construct representations (including root of 1 representations) for $pB_q(n) = U_q[osp(1/2n)]$. To this end one can use n-pairs of deformed pB operators as given in [2, 4, 5]. The solution, however, is not going to be easy for arbitrary values of p, if one takes into account that the problem has not been solved even in the non-deformed case. Only the case with $p = 1$ is easy. It leads directly to root of 1 representations of $U_q[osp(1/2n)]$, if one uses q-commuting deformed Bose operators as defined in [5]. Other root of 1 representations based on a realization with commuting q-Bose operators (which means also the case $p = 1$) were obtained in [44]. In this relation we note that n pairs of commuting deformed Bose operators are already generators of $U_q[osp(1/2n)]$ (in the q-Bose representation). Therefore they provide the simplest q-Boson realization of $U_q[osp(1/2n)]$ [28].

Finally we mention that all our representations correspond to q being an even root of unity: $q^{4k} = 1$. In the case of deformed simple Lie algebras this seems to be the more difficult case. Complete results only exist for q being odd roots of 1 [45].

Acknowledgments

The authors would like to thank Professor Randjbar-Daemi for the kind hospitality at the High Energy section of ICTP, Trieste. Constructive discussions with Professors J Van der Jeugt and R Jagannathan are greatly acknowledged. The work was supported by the Grant Φ-416 of the Bulgarian Foundation for Scientific Research.

References

[1] Celeghini E, Palev T D and Tarlini M 1990 *Preprint* YITP/K-865, Kyoto; 1991 *Mod. Phys. Lett.* **5B** 187
[2] Palev T D 1993 *J. Phys. A: Math. Gen.* 26 L1111
[3] Hadjiivanov L K 1993 *J. Math. Phys.* 34 5476
[4] Palev T D 1994 Quantization of the Lie algebra $so(2n+1)$ and of the Lie superalgebra $osp(1/2n)$ with preoscillator generators *Preprint* TWI-94-29, University of Ghent; *J. Group Theory Phys.* 3 to appear
[5] Palev T D and Van der Jeugt J 1995 *J. Phys. A: Math. Gen.* 28 2605
[6] Palev T D 1994 *Lett. Math. Phys.* 31 151
[7] Greenberg O W and Mohapatra R N 1987 *Phys. Rev. Lett.* 59 2507
[8] Floreanini R and Vinet L 1990 *J. Phys. A: Math. Gen.* 23 L1019

[9] Odaka K, Kishi T and Kamefuchi S 1991 *J. Phys. A: Math. Gen.* **24** L591
[10] Beckers J and Debergh N 1991 *J. Phys. A: Math. Gen.* **247** L1277
[11] Chaturvedi S and Srinivasan V 1991 *Phys. Rev.* A **44** 8024
[12] Krishna-Kumari M, Shanta P, Chaturvedi S and Srinivasan V 1992 *Mod. Phys. Lett.* **7A** 2593
[13] Bonatsos D and Daskaloyannis C 1993 *Phys. Lett.* **307B** 100, and the references therein
[14] Flato M, Hadjiivanov L K and Todorov I T 1993 *Found. Phys.* **23** 571
[15] Macfarlane A J 1993 Generalized oscillator systems and and their parabosonic interpretation *Preprint* DAMPT 93-37
[16] Palev T D 1993 *J. Math. Phys.* **34** 4872
[17] Quesne C 1994 *Phys. Lett.* **193A** 245
[18] Van der Jeugt J and Jagannathan R 1994 Polynomial deformations of $osp(1/2)$ and generalized parabosons hep-th/9410145; *J. Phys. A: Math. Gen.* to appear
[19] Macfarlane A J 1994 *J. Math. Phys.* **35** 1054
[20] Cho K H, Chaiho Rim, Soh D S and Park S U 1994 *J. Phys. A: Math. Gen.* **27** 2811
[21] Chakrabarti R and Jagannathan R 1994 *J. Phys. A: Math. Gen.* **27** L277; 1994 *Int. J. Mod. Phys.* A **9** 1411
[22] Green H S 1994 *Aust. J. Phys.* **47** 109
[23] Ganchev A and Palev T D 1978 *Preprint* JINR P2-11941; 1980 *J. Math. Phys.* **21** 797
[24] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[25] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[26] Sun C P and Fu H C 1989 *J. Phys. A: Math. Gen.* **22** L983
[27] Hayashi T 1990 *Commun. Math. Phys.* **127** 129
[28] Palev T D 1993 *Lett. Math. Phys.* **28** 187
[29] Palev T D 1994 *J. Phys. A: Math. Gen.* **27** 7373
[30] Chaichan M and Kulish P 1990 *Phys. Lett.* **234B** 72
[31] Bracken A J, Gould M D and Zhang R B 1990 *Mod. Phys. Lett.* **5A** 331
[32] Floreanini R, Spiridonov V P and Vinet L 1990 *Phys. Lett.* **242B** 383
[33] Floreanini R, Spiridonov V P and Vinet L 1991 *Commun. Math. Phys.* **137** 149
[34] d'Hoker, Floreanini R and Vinet L 1991 *J. Math. Phys.* **32** 1247
[35] Khoroshkin S M and Tolstoy V N 1991 *Commun. Math. Phys.* **141** 599
[36] Kulish P P and Reshetikhin N Yu 1989 *Lett. Math. Phys* **18** 143
[37] Saleur H 1990 *Nucl. Phys.* B **336** 363
[38] Sun Chang-Pu, Fu Hong-Chen and Ge Mo-Lin 1991 *Lett. Math. Phys* **23** 19
[39] Ge Mo-Lin, Sun Chang-Pu and Xue Kang 1992 *Phys. Lett.* **163A** 176
[40] Sun Chang-Pu 1993 *Nuovo Cimento* B **108** 499
[41] Ubriaco M R 1993 *Mod. Phys. Lett.* **8A** 89
[42] Green H S 1953 *Phys. Rev.* **90** 270
[43] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523
[44] Fu Hong-Chen and Ge Mo-Lin 1992 *Commun. Theor. Phys.* **18** 373
[45] De Concini C and Kac V G 1990 Colloque dixmier *Progr. Math.* **92** 471